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FACTORIZATIONS, M-SEPARATION, AND EXTREMAL-EPIREFLECTIVE SUBCATEGORIES

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Every factorization structure on certain concrete categories induces an extremal-epireflective subcategory. We study these subcategories, give conditions under which an extremal-epireflective subcategory can be so induced, and then show that there are extremal-epireflective subcategories that cannot be induced in this manner.

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Introduction

Let (E, M) be a factorization structure for single morphisms on a 'nice' category $\mathscr{C}$. In this paper we look at the relation between (E, M) factorization structures on $\mathscr{C}$ and extremal-epireflective subcategories of $\mathscr{C}$.

Preliminaries are given in Section 1.

In Section 2, we study $\mathscr{C}_M$, the full subcategory of $\mathscr{C}$ whose objects are those objects X in $\mathscr{C}$ whose diagonal $\Delta_X : X \to X^2 \in M$. $\mathscr{C}_M$ is an extremal-epireflective subcategory of $\mathscr{C}$, and has the property that (E, M) induces a factorization structure (E', M') on $\mathscr{C}_M$ such that $E' \subseteq \mathrm{Epi}$ (in $\mathscr{C}_M$).

This gives rise to the following question: If $\mathscr{A}$ is extremal-epireflective in $\mathscr{C}$, does $\mathscr{A} = \mathscr{C}_M$ for some factorization structure (E, M) on $\mathscr{C}$? In Section 3, we give conditions on $\mathscr{A}$ and $\mathscr{C}$ in order that the answer be affirmative. (See Corollary 3.8.)

In Section 4 we give an example of a subcategory of **Top** for which these conditions are satisfied. We then show that **FH**, the full subcategory of **Top** whose objects consist of the functionally Hausdorff spaces, does not satisfy the conditions of Corollary 3.8, and, furthermore, there is no factorization structure (E, M) on **Top** for which $\mathbf{FH} = \mathbf{Top}_M$.

1. Preliminaries

$\mathscr{C}$ is a complete, well-powered concrete category with forgetful functor $T : \mathscr{C} \to \mathbf{Set}$ that preserves monomorphisms.

Definition 1.1 [6]. Let E and M be classes of morphisms of $\mathscr{C}$. (E, M) is a *factorization structure* for $\mathscr{C}$ iff

(1) E and M are closed under composition;

(2) $E \cap M$ contains all isomorphisms;

(3) Each morphism f in $\mathscr{C}$ is factorizable; i.e., $f = me$, where $e \in E$ and $m \in M$;

(4) $\mathscr{C}$ has the (E, M) unique diagonalization property; i.e., if $mf = ge$, with $m \in M$, $e \in E$, there exists a unique morphism d that makes the following diagram commute.

$$\begin{tikzcd} {} \arrow[r, "e"] \arrow[d, "f"'] & {} \arrow[d, "g"] \arrow[dl, dashed, "d"'] \\ {} \arrow[r, "m"'] & {} \end{tikzcd}$$

If (E, M) is a factorization structure for $\mathscr{C}$, $\mathscr{C}$ will be called an (E, M) *category.*

Definition 1.2 [6, 7, 12]. A monomorphism is an *embedding* iff it is a T-initial lifting of a monomorphism. We denote the class of embeddings by M_0.

Properties and examples of embeddings can be found in [6] and [7].

Definition 1.3 [8]. If (E, M) is a factorization structure for $\mathscr{C}$ such that each m in M is an embedding, (E, M) will be called a *strong factorization structure.*

For the remainder of this article, we make the following additional assumption on the category $\mathscr{C}$: There exists a class of epimorphisms E_0 such that $\mathscr{C}$ is an (E_0, M_0) category.

The notion of hull operator was motivated by the definitions of function space convex hull, and of limit operator [3]. A complete discussion, with examples, may be found in [6] and [7].

Definition 1.4 [8]. A *hull operator* on $\mathscr{C}$ is an operator Q that assigns to each M_0-subobject (S, i_s) of every object X an M_0-subobject of X called the *XQ hull of* (S, i_s), denoted $(XQ(S, i_s), j_s)$, which satisfies the following conditions:

(1) There exists $i' : S \to XQ(S, i_s) \in M_0$ such that $j_s i' = i_s$;

(2) If $i_A : A \to B$, $i_B : B \to X$, and $i'_A : A \to X$ with $i'_A = i_B i_A$, then there exists $j' \in M_0$ such that $j_A = j_B j'$, where $j_B : XQ(B, i_B) \to X$, $j_A : XQ(A, i'_A) \to X$, and $j' : XQ(A, i'_A) \to XQ(B, i_B)$;

(3) If $i_A : A \to Y$, $i_Y : Y \to X \in M_0$, $i_Y = j_Y$, then $j_A = i_Y j'_A$, where $j_A : XQ(A, i_Y i_A) \to X$, $j'_A : YQ(A, i_A) \to Y$;

(4) If $f : X \to Y$ is a morphism, $i_z : Z \to Y \in M_0$, $j_z : YQ(Z, i_z) \to Y$, $i_z = j_z$, and the following square is a pullback

$$\begin{tikzcd} E \arrow[r, "p_2"] \arrow[d, "p_1"'] & X \arrow[d, "f"] \arrow[dl, dashed] \\ Z \arrow[r, "i_z"'] & Y \end{tikzcd}$$

then $p_2 = j_E$, where $j_E : XQ(E, p_2) \to X$;

(5) If $(T, i_T) = (XQ(S, i_s), j_s)$, then $i_T = j_T = j_s$, where $j_T : XQ(T, i_T) \to X$.

Definition 1.5 [8]. A *hull subobject* operator on $\mathscr{C}$ is an operator R that assigns to each object X a class of M_0-subobjects of X, denoted $R(X)$, which satisfies the following conditions:

(1) $R(X)$ is closed under intersections;

(2) If $f : X \to Y$ is a morphism, $(S, i_s) \in R(Y)$, and the following square is a pullback

$$\begin{tikzcd} E \arrow[r, "p_2"] \arrow[d, "p_1"'] & X \arrow[d, "f"] \arrow[dl, dashed] \\ S \arrow[r, "i_s"'] & Y \end{tikzcd}$$

then $(E, p_2) \in R(X)$;

(3) If $(Y, i_Y) \in R(X)$, then $(S, i_s) \in R(Y)$ iff $(S, i_Y i_s) \in R(X)$;

(4) $(X, \mathrm{id}) \in R(X)$.

The following theorem shows the relationship between strong factorization structures, hull operators, and hull subobject operators.

Theorem 1.6 [8]. *There exists a* 1-1 *correspondence between the following* 3 *families*:

(1) *The class of strong factorization structures on* $\mathscr{C}$;

(2) *The class of hull operators on* $\mathscr{C}$;

(3) *The class of hull subobject operators on* $\mathscr{C}$.

Definition 1.7 [4, 13]. Let $\mathscr{A}$ be a subcategory of $\mathscr{C}$. $f : X \to Y$ is an *$\mathscr{A}$-epi* iff for all objects A in $\mathscr{A}$, $rf = sf$ implies $r = s$, where $r, s : Y \to A$.

Definition 1.8 [4]. $\mathscr{C}$ is *$(E, \mathscr{A})$-co-well-powered* iff every object X in $\mathscr{C}$ has a representative set of $(E, \mathscr{A})$ morphisms, where an $(E, \mathscr{A})$ morphism is a morphism in the class E with codomain in ob $\mathscr{A}$.

Definition 1.9. $\mathcal{A}$ is *M-hereditary iff* $m: X \to A \in M$, A an object in $\mathcal{A}$ implies X an object in $\mathcal{A}$.

Definition 1.10 [6]. The *diagonal* $\Delta_x : X \to X^2$ is the unique morphism defined by the categorical product $(\prod_{i=1,2} X_i, (\pi_i))$, where $X_i = X$, $i = 1, 2$; i.e. if $\pi_i : X^2 \to X$, then Δ_x is the unique morphism such that the following diagram commutes.

$$\begin{tikzcd}
 & X \\
X \arrow[ur, "\mathrm{id}"] \arrow[r, dashed, "\Delta_x"] \arrow[dr, "\mathrm{id}"'] & X^2 \arrow[u, "\pi_1"'] \arrow[d, "\pi_2"] \\
 & X
\end{tikzcd}$$

2. Factorizations and diagonals

Suppose that (E, M) is a factorization structure (for single morphisms) on $\mathcal{C}$ such that $E \not\subseteq$ Epi. In this section we study the largest subcategory $\mathcal{Y}$ of $\mathcal{C}$ that has the property that (E, M) induces a factorization structure (E', M') on $\mathcal{Y}$, and $E' \subseteq \mathcal{Y}$-epi.

The following definition was motivated by the fact that a topological space is Hausdorff iff its diagonal is closed.

Definition 2.1. Let (E, M) be a factorization structure on $\mathcal{C}$, and $X \in \text{ob}\,\mathcal{C}$. X will be called M-separated iff $\Delta_x \in M$.

Definition 2.2. $\mathcal{C}_M$ is the full subcategory of $\mathcal{C}$ whose objects, denoted ob $\mathcal{C}_M$, are all the M-separated objects in $\mathcal{C}$, where (E, M) is a factorization structure on $\mathcal{C}$.

The basic properties of $\mathcal{C}_M$ which appear below are due to Herrlich [4]. The subcategory $\mathcal{C}_M$ is the same as the subcategory E-Sep which appears in Herrlich, Salicrup, and Strecker [5].

Examples. (1) $\mathcal{C} = \mathbf{Top}$, E is the class of dense maps, M is the class of closed embeddings. $\mathbf{Top}_M$ is **Haus**.

(2) $\mathcal{C} = \mathbf{Top}$, E is the class of front dense maps [1], M is the class of front closed embeddings, $\mathbf{Top}_M = \mathbf{Top}_0$, whose objects are the T_0 spaces. (A is front dense in B iff for each $b \in B$ there is a neighborhood V of b such that $A \cap V \cap \text{cl}\{b\} \neq \phi$.)

(3) $\mathcal{C} = \mathbf{Top}$, E is the class of i-dense maps, M is the class of i-closed embeddings. $\mathbf{Top}_M = \mathbf{Top}_0$. ($A$ is i-dense in B iff for each $b \in B$, there exists $a \in A$ so that $\{a, b\}$ is indiscrete in B.)

(4) $\mathscr{C} = \mathbf{Top}$, E is the class of c-dense maps, M is the class of c-closed embeddings. $\mathbf{Top}_M = \mathbf{Top}_1$, whose objects are the T_1 spaces. (A is c-dense in B iff for each $b \in B$, there exists $a \in A$ such that $a \in \mathrm{cl}\{b\}$.)

(5) $\mathscr{C} = \mathbf{Top}$, E is the class of r-dense maps, M is the class of r-closed embeddings. $\mathbf{Top}_M = \mathbf{R}_0$, whose objects are the symmetric spaces. (A is r-dense in B iff for each $b \in B$, there exists $a \in A$ such that $a \in \mathrm{cl}\{b\}$ and $b \in \mathrm{cl}\{a\}$.)

In the following examples, (E, M) is the factorization structure induced by the given hull operator Q. In order to describe the induced factorization structure we need the following Definition.

Definition 2.3. $f: X \to Y$ is a morphism in $\mathscr{C}$. The *image of f*, denoted Im(f), is the subobject (C, m) of Y, where $f = me$ is the (E_0, M_0) factorization of f. $\mathrm{Im}(f)$ may also be denoted by either $(f(X), m)$ or, more simply, $f(X)$.

Each hull operator Q induces a unique factorization structure (Theorem 1.6) as follows:

E is the class of Q-dense morphisms; i.e., $f: X \to Y \in E$ iff $YQ(f(X), m) = (Y, \mathrm{id})$;

M is the class of Q closed embeddings; i.e., $f: X \to Y \in M$ iff $f \in M_0$ and $YQ(f(X), m) = (f(X), m)$.

(6) $\mathscr{C} = \mathbf{Top}$, $XQ(A) = \{x \in X$ such that every clopen neighborhood of x in X meets $A\}$. ob $\mathbf{Top}_M$ is the class of spaces whose quasicomponents are singletons.

(7) $\mathscr{C} = \mathbf{Top}$, $XQ(A)$ is the union of all components of elements of A. ob $\mathbf{Top}_M$ is the class of totally disconnected spaces.

(8) $\mathscr{C} = \mathbf{Top}$, Q is the hull operator with the following property: $S = XQ(S)$ iff for each $A \subseteq S$, card $A < k$, a regular infinite cardinal, $\mathrm{cl}_x A \subseteq S$. ob $\mathbf{Top}_M$ is the class of spaces X such that for all $x, y \in X$, $x \neq y$ and for each $A \subseteq X$, card $A < k$, there exists neighborhoods U of x, V of y such that $U \cap V \cap A \neq \phi$.

(9) $\mathscr{C} = \mathbf{Funsp}$, [7], **Funsp** is the category whose objects are all pairs (X, H), where X is a topological space and H is a linear subspace of $C(X)$, the continuous real-valued functions on X, that contains the constants; the morphisms $f: (X, H) \to (Y, K)$ are all continuous maps $f: X \to Y$ such that $Kf \subseteq H$. If Q is topological closure, ob $\mathscr{C}_M$ consists of all pairs (X, H) with X Hausdorff.

(10) $\mathscr{C} = \mathbf{Funsp}$, Q is closed convex hull. ob $\mathscr{C}_M$ consists of all pairs (X, H) where X is Hausdorff and H separates the points of X. ($(X, H)Q(A) = \{x \in X$ such that $h(x) \leq \sup h(A)$ for all $h \in H.\}$)

(11) $\mathscr{C} = \mathbf{Funsp}$, Q is closed affine hull. $\mathscr{C}_M$ is the subcategory obtained in example (10). ($(X, H)Q(A) = \{x \in X$ such that $h(A) = 0 \Rightarrow h(x) = 0$ for all $h \in H.\}$)

(12) $\mathscr{C}$ is the category of Abelian groups and group homomorphisms, Q is the isolator ([2] and [14]). $\mathscr{C}_M$ is the full subcategory whose objects are the torsion-free Abelian groups. ($GQ(S) = \{x \in G$ such that $nx \in S$ for some positive integer $n.\}$)

(13) $\mathscr{C}$ is the category of (compact Hausdorff)-generated spaces, Q is topological closure. ob $\mathscr{C}_M$ is the class of weakly Hausdorff spaces [10, 16].

(14) $\mathscr{C} = \mathbf{Pos}$, the category of partially ordered sets and monotone functions, Q is μ-closure. ob $\mathscr{C}_M$ consists of the objects $(X, \leqslant)$ where $\leqslant$ is equality. (A is μ-dense in B iff for each $b \in B$ there exists $a \in A$ such that $a \leqslant b$.)

Lemma 2.4. *$\mathscr{C}_M$ is M-hereditary.*

Proof. Let $Y \in \mathscr{C}_M$, $m : X \to Y \in M$.

$$\begin{tikzcd} X \arrow[r, "m"] \arrow[d, "\Delta_X"'] & Y \arrow[d, "\Delta_Y"] \\ X^2 \arrow[r, "m^2 = m \times m"'] & Y^2 \end{tikzcd}$$

$m \in M \Rightarrow m^2 \in M$. $\Delta_Y m = m^2 \Delta_X \in M$. Thus $\Delta_X \in M$. $\square$

Lemma 2.5. *$\mathscr{C}_M$ is mono-hereditary.*

Proof. Consider the following pullback diagram:

$$\begin{tikzcd} X \arrow[dr, dashed, "h"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "\Delta_X"'] & & \\ & E \arrow[r, "p_1"] \arrow[d, "p_2"'] & Y \arrow[d, "\Delta_Y"] \\ & X^2 \arrow[r, "f^2"'] & Y^2 \end{tikzcd}$$

Recall f^2 is defined by

$$\begin{tikzcd} X^2 \arrow[r, "f^2"] \arrow[d, "\pi_i"'] & Y^2 \arrow[d, "\pi_i'"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

where $\pi_i(\pi_i')$ is the projection onto $X(Y)$. h is an isomorphism since

$$f\pi_1 p_2 h = \pi_1' f^2 p_2 h = \pi_1' \Delta_Y f = f$$

and

$$p_1 h \pi_1 p_2 = f \pi_1 p_2 = \pi_1' f^2 p_2 = \pi_1' \Delta_Y p_1 = p_1.$$

If f is a monomorphism, then f^2 and p_1 are also monomorphisms.

$$\pi_1 p_2 h = \mathrm{id}_X \quad \text{and} \quad h \pi_1 p_2 = \mathrm{id}_E.$$

Thus $\Delta_Y \in M$ implies $\Delta_X \in M$. $\square$

Lemma 2.6. *$\mathscr{C}_M$ is closed under products in $\mathscr{C}$.*

Proof. Let $X_i \in \mathscr{C}_M$

$$\begin{tikzcd} \Pi X_i \arrow[r, "\pi\Delta X_i"] \arrow[d, "p_i"'] & (\Pi X_i)^2 = \Pi(X_i)^2 \arrow[d, "q_i"] \\ X_i \arrow[r, "\Delta_{X_i}"'] & X_i^2 = X_i^2 \end{tikzcd}$$

$\Pi\Delta_{X_i} = \Delta_{\Pi X_i}$. Thus $\Delta_{\Pi X_i} \in M$ since M is closed under products. $\square$

Corollary 2.7. *$\mathscr{C}_M$ is extremal-epireflective in $\mathscr{C}$.*

Lemma 2.8. *(E, M) induces a factorization structure (E', M') on $\mathscr{C}_M$.*

Proof. Let $f: X \to Y$, $X, Y \in \mathscr{C}_M$. $f = me$, $e: X \to Z \in E$, $m: Z \to Y \in M$. $Z \in \text{ob}\, \mathscr{C}_M$ since $\mathscr{C}_M$ is M-hereditary. Thus $E' = \{e: X \to Z \mid X, Z \in \mathscr{C}_M, e \in E\}$ and $M' = \{m: Z \to Y, Y \in \mathscr{C}_M, m \in M\}$. $\square$

Lemma 2.9. $E \subseteq \{\mathscr{C}_M\text{-epi}\}$.

Proof. Suppose $e: X \to Z \in E$, $r, s: Z \to Y$, $Y \in \text{ob}\, \mathscr{C}_M$, and $re = se$. Let $h = re$. Then there exists d such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[d, "h"'] & Z \arrow[d, "\langle r, s\rangle"] \arrow[dl, dashed, "d"'] \\ Y \arrow[r, "\Delta_Y"'] & Y^2 \end{tikzcd}$$

Thus

$$r = p_1\langle r, s\rangle = p_1\Delta_Y d = d \quad \text{and} \quad s = p_2\langle r, s\rangle = p_2\Delta_Y d = d,$$

where p_1, p_2: $Y^2 \to Y$ are the projections. $\square$

Lemma 2.10. *$\mathscr{C}_M$ is the largest full subcategory $\mathscr{Y}$ of $\mathscr{C}$ such that $E \subseteq \{\mathscr{Y}\text{-epi}\}$.*

Proof. Suppose $\mathscr{Y}$ is a full subcategory of $\mathscr{C}$ with $E \subseteq \{\mathscr{Y}\text{-epi}\}$. Let $Y \in \text{ob}\, \mathscr{Y}$. Let $\Delta_Y: Y \to Y^2 = me$, $e: Y \to X \in E$, $m: X \to Y^2 \in M$, and $p_i: Y^2 \to Y$ be the projections. Then $p_1 me = p_1\Delta_Y = \text{id}_Y = p_2\Delta_Y = p_2 me$. Then $p_1 m = p_2 m (= n)$, $ne = p_1 me = p_1 \Delta_Y = \text{id}_Y$, $\Delta_Y ne = \Delta_Y = me$ implies $\Delta_{Y} n = m$, and the following diagram commutes:

$$\begin{tikzcd} Y \arrow[r, "e"] \arrow[d, "\text{id}_Y"'] & X \arrow[d, "m"] \arrow[dl, dashed, "n"'] \\ Y \arrow[r, "\Delta_Y"'] & Y^2 \end{tikzcd}$$

$ne = \text{id}_Y$, and $ene = ep_1 me = ep_1\Delta_Y = e$ implies $en = \text{id}_X$. Thus $e = n^{-1}$, $\Delta_Y \in M$ and $Y \in \text{ob}\, \mathscr{C}_M$. Hence $\mathscr{Y} \subseteq \mathscr{C}_M$. $\square$

Proposition 2.11. *$\mathscr{C}_M$ is the largest full subcategory $\mathscr{Y}$ of $\mathscr{C}$ such that the following hold:*

(a) *$\mathscr{Y}$ is M-hereditary (and thus (E, M) induces a factorization structure (E_Y, M_Y) on $\mathscr{Y}$.);*

(b) *$\mathscr{Y}$ is closed under products in C;*

(c) $E_Y \subseteq \mathscr{Y}$-epi.

Furthermore, if $\mathscr{C}$ is $(E, \mathscr{C}_M)$-co-well-powered, then $\mathscr{C}_M$ is the largest E-reflective full subcategory $\mathscr{Y}$ of $\mathscr{C}$ such that $e \in E_Y$ implies e is an epimorphism in $\mathscr{Y}$.

Proof. (a), (b), and (c) follow from Lemma 2.10. Suppose that $\mathscr{C}$ is $(E, \mathscr{Y})$-co-well-powered with $e \in E_Y \Rightarrow e$ epi in $\mathscr{Y}$. We will show that $\mathscr{C}_M$ is E-reflective. If $\mathscr{Y}$ is also E-reflective and satisfies the hypothesis, then $\mathscr{Y}$ satisfies (a), (b), and (c) of this proposition, and hence $\mathscr{Y} \subseteq \mathscr{C}_M$.

Let $X \in \mathrm{ob}\, \mathscr{C}$; $\{e_i : X \to A_i\}$ be a representative set of all E-morphisms with codomain in ob $\mathscr{C}_M$. $\langle e_i \rangle : X \to \Pi Ai$. Let $\langle e_i \rangle = me$, $e : X \to Y \in E$, $m : Y \to \Pi Ai \in M$. Then $e : X \to Y$ is a $\mathscr{C}_M$-reflection. That e is the reflection follows from the fact that $Y \in \mathrm{ob}\, \mathscr{C}_M$ (since $\mathscr{C}_M$ is productive and M-hereditary), e is obviously $\mathscr{C}_M$-extendable, and the extension is unique since $e \in \mathscr{C}_M$-epi.

$\mathscr{C}_M$ is E-reflective, and by the remark in the preceding paragraph, it is the largest E-reflective subcategory $\mathscr{Y}$ of $\mathscr{C}$ with $E_Y \subseteq \mathscr{Y}$-epi. □

Thus, if (E, M) is a factorization structure on $\mathscr{C}$, there is a largest full subcategory $\mathscr{C}_M$, which is E-reflective in $\mathscr{C}$ if $\mathscr{C}$ is $(E, \mathscr{C}_M)$-co-well-powered. We have seen (Corollary 2.7) that $\mathscr{C}_M$ is extremal-epireflective in $\mathscr{C}$.

In the next section we investigate the following question: If $\mathscr{A}$ is extremal-epi reflective in $\mathscr{C}$, does there exist a factorization structure (E, M) on $\mathscr{C}$ such that $\mathscr{A} = \mathscr{C}_M$?

3. Extremal-epireflective subcategories and diagonals

Let $\mathscr{A}$ be a full extremal-epireflective subcategory of $\mathscr{C}$. We now look at those conditions that $\mathscr{A}$ must satisfy in order that $A = \mathscr{C}_M$ for some factorization structure (E, M) on $\mathscr{C}$.

Definition 3.1. Let $i : S \to X$ such that there exist $r, s : X \to A$, $A \in \mathrm{ob}\, \mathscr{A}$, with $i = \mathrm{Equ}(r, s)$. Then i will be called an *$\mathscr{A}$-regular morphism.*

Let M denote the class of $\mathscr{A}$-regular morphisms. Define $R(X)$, for each $X \in \mathscr{C}$, as follows: $R(X) = \{(S, i) : i : S \to X \in M.\}$

We have the following:

Lemma 3.2. *If M is closed under composition, then R is a hull subobject operator.*

Proof. We will show that the conditions of Definition 1.5 are satisfied.

(1) Suppose $m_i : S_i \to X$, where $m_i = \mathrm{equ}(r_i, s_i)$, $s_i, r_i : X \to A_i$. Let $r = \langle r_i \rangle : X \to \Pi A_i$, $s = \langle s_i \rangle : X \to \Pi A_i$. Then $(C, d) = \mathrm{Equ}(r, s)$ iff $(C, d) = \cap (S_i, m_i)$.

(2) Suppose the following square is a pullback:

$$\begin{tikzcd} E \arrow[r, "p_2"] \arrow[d, "p_1"'] & X \arrow[d, "f"] \\ S \arrow[r, "m"'] & Y \end{tikzcd}$$

where $(S, m) = \mathrm{Equ}(r, s)$, $r, s: Y \to A$. We will show that $(E, p_2) = \mathrm{Equ}(rf, sf)$. Consider

$$\begin{tikzcd} E \arrow[r, "p_2"] & X \arrow[r, shift left, "rf"] \arrow[r, shift right, "sf"'] & A \\ T \arrow[ur, "k"'] & & \end{tikzcd}$$

$rfp_2 = rmp_1 = smp_1 = sfp_2$. If $rfk = sfk$, then, since $(S, m) = \mathrm{Equ}(r, s)$, there exists j such that

$$\begin{tikzcd} S \arrow[r, "m"] & Y \arrow[r, shift left, "r"] \arrow[r, shift right, "s"'] & A \\ T \arrow[u, dashed, "j"] \arrow[ur, "fk"'] & & \end{tikzcd}$$

commutes.

Consider the pullback

$$\begin{tikzcd} T \arrow[dr, dashed, "i"] \arrow[drr, bend left, "k"] \arrow[ddr, bend right, "j"'] & & \\ & E \arrow[r, "p_2"] \arrow[d, "p_1"] & X \arrow[d, "f"] \\ & S \arrow[r, "m"'] & Y \end{tikzcd}$$

Thus $p_2 i = k$, so that $rfp_2 i = sfp_2 i$, and $(E, p_2) = \mathrm{Equ}(rf, sf)$.

That conditions (3) and (4) are satisfied is immediate. □

Remark. The relationship between $\mathscr{A}$-regular morphisms and closure operators was obtained by S. Salbany [15] for subcategories of **Top**.

Lemma 3.3. *$m \in M$ if and only if m is a pullback of (B, e), where $B \in \mathrm{ob}\, \mathscr{A}$ and e is an $\mathscr{A}$-regular morphism.*

Proof. If m is a pullback of an $\mathscr{A}$-regular morphism, then $m \in M$. (See part 2 of Lemma 3.2.)

Suppose $m: X \to Y \in M$, $X \notin \operatorname{ob} \mathscr{A}$. Then $m = \operatorname{equ}(r, s)$ where $r, s: Y \to A$, $A \in \operatorname{ob} \mathscr{A}$. Let r_Y denote the reflection map $Y \to \hat{Y}$ into $\mathscr{A}$. Then there exist $\hat{r}, \hat{s}$ such that the following commute.

$$\begin{tikzcd} Y \arrow[r, "r_Y"] \arrow[d, "r"'] & Y \arrow[dl, dashed, "\hat{r}"] \\ A & \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r, "r_Y"] \arrow[d, "s"'] & Y \arrow[dl, dashed, "\hat{s}"] \\ A & \end{tikzcd}$$

Let $e: B \to Y = \operatorname{Equ}(r, s)$. Then the following square is a pullback

$$\begin{tikzcd} E \arrow[dr, dashed, "\gamma"] \arrow[drr, bend left, "\alpha"] \arrow[ddr, bend right, "\beta"'] & & \\ & X \arrow[r, "m"] \arrow[d, "h"] & Y \arrow[d, "r_Y"] \\ & B \arrow[r, "e"'] & \hat{Y} \end{tikzcd}$$

where h is defined by

$$\begin{tikzcd} B \arrow[r, "e"] & \hat{Y} \arrow[r, shift left, "\hat{r}"] \arrow[r, shift right, "\hat{s}"'] & A \\ & X \arrow[ul, dashed, "h"] \arrow[u, "r_Y m"'] & \end{tikzcd}$$

and γ is defined by

$$\begin{tikzcd} X \arrow[r, "m"] & Y \arrow[r, shift left, "r"] \arrow[r, shift right, "s"'] & A \\ & E \arrow[ul, dashed, "\gamma"] \arrow[u, "\alpha"'] & \end{tikzcd}$$

$eh\gamma = r_Y m\gamma = r_Y\alpha = e\beta$. Since e is a monomorphism, $h\gamma = \beta$. $\square$

Lemma 3.4. *Let $\mathscr{A}$ be an extremal-epireflective subcategory of $\mathscr{C}$, and $X \in \operatorname{ob} \mathscr{A}$. If $r_X\alpha = r_X\beta$, then $f\alpha = f\beta$ for all $f: X \to A$, $A \in \operatorname{ob} \mathscr{A}$.*

Proof. Since r_X is the reflection $r_X: X \to \hat{X}$, we have

$$\begin{tikzcd} X \arrow[r, "r_X"] \arrow[d, "f"'] & \hat{X} \arrow[dl, dashed, "\hat{f}"] \\ A & \end{tikzcd}$$

where $f = \hat{f} r_X$. Thus $f\alpha = \hat{f} r_X \alpha = \hat{f} r_X \beta = f\beta$. $\square$

Lemma 3.5. *Let $\mathscr{C}$ be a topological category, $\mathscr{A}$ an extremal-epireflective subcategory, and $X \notin \mathrm{ob}\,\mathscr{A}$. Then there exist constants α, β such that $r_x\alpha = r_x\beta$.*

Proof. Since $X \notin \mathrm{ob}\,\mathscr{A}$, r_x is not a mono. Therefore $\mathrm{Tr}_x : TX \to T\hat{X}$ is not a mono in **Set**. Thus, there exist $\alpha', \beta' : \bar{Z} \to TX$ such that $\mathrm{Tr}_x\,\alpha' = \mathrm{Tr}_x\,\beta'$, $\alpha' \neq \beta'$, and α', β' constants with singleton domain. (α', β' constants means $\alpha'(\bar{Z})$ and $\beta'(\bar{Z})$ are singletons.) Let $\alpha, \beta : Z \to X$ be T-initial liftings of α' and β' respectively. Then α and β are constants. □

Theorem 3.6. *Let $\mathscr{C}$ be a connected topological category, and $\mathscr{A}$ an extremal-epireflective subcategory. If $X \in \mathrm{ob}\,\mathscr{A}$, then $\Delta_x \in M$, where M is the class of all $\mathscr{A}$-regular morphisms.*

Proof. Let $\langle \mathrm{id}, \bar{\beta}\rangle$ and i be defined by

$$\begin{tikzcd} & X & \\ X \arrow[ur, "\mathrm{id}"] \arrow[r, "\langle \mathrm{id}, \bar{\beta}\rangle"] \arrow[dr, "\bar{\beta}"'] & X \times Z \arrow[u, "p_1"'] \arrow[d, "p_2"] \\ & Z & \end{tikzcd} \qquad \begin{tikzcd} & X & \\ X \times Z \arrow[ur, "p_1"] \arrow[r, "i"] \arrow[dr, "\beta p_2"'] & X^2 \arrow[u, "\pi_1"'] \arrow[d, "\pi_2"] \\ & X & \end{tikzcd}$$

where $\bar{\beta} : X \to Z$. ($\bar{\beta}$ exists since $\mathscr{C}$ is connected.) Suppose that $(X, \Delta_x) = \mathrm{Equ}(f, g)$, where $f, g : X^2 \to A$, $A \in \mathrm{ob}\,\mathscr{A}$.

Consider the morphism $i\langle \mathrm{id}, \bar{\beta}\rangle\beta$. $\pi_1 i\langle \mathrm{id}, \bar{\beta}\rangle\beta = p_1\langle \mathrm{id}, \bar{\beta}\rangle = \mathrm{id}\,\beta = \beta$ and $\pi_2 i\langle \mathrm{id}, \bar{\beta}\rangle\beta = \beta p_2\langle \mathrm{id}, \bar{\beta}\rangle\beta = \beta\bar{\beta}\beta = \beta$. Thus, $i\langle \mathrm{id}, \bar{\beta}\rangle\beta = \Delta_x\beta$. Since $f\Delta_x = g\Delta_x$, $f\Delta_x\beta = g\Delta_x\beta$, and thus $fi\langle \mathrm{id}, \bar{\beta}\rangle\beta = gi\langle \mathrm{id}, \bar{\beta}\rangle\beta$. $fi\langle \mathrm{id}, \bar{\beta}\rangle : X \to A$, $A \in \mathrm{ob}\,\mathscr{A}$. Therefore $fi\langle \mathrm{id}, \bar{\beta}\rangle\beta = fi\langle \mathrm{id}, \bar{\beta}\rangle\alpha$ and also $gi\langle \mathrm{id}, \bar{\beta}\rangle\alpha = gi\langle \mathrm{id}, \bar{\beta}\rangle\beta$. This implies that $fi\langle \mathrm{id}, \bar{\beta}\rangle\alpha = gi\langle \mathrm{id}, \bar{\beta}\rangle\alpha$. Since $(X, \Delta_x) = \mathrm{Equ}(f, g)$, there exists $h : Z \to X$ with $\Delta_x h = i\langle \mathrm{id}, \bar{\beta}\rangle\alpha$. It follows that $\pi_1\Delta_x h = \pi_2\Delta_x h$. However, $\pi_1 i\langle \mathrm{id}, \bar{\beta}\rangle\alpha = p_1\langle \mathrm{id}, \bar{\beta}\rangle\alpha = \mathrm{id}\,\alpha = \alpha$, and $\pi_2 i\langle \mathrm{id}, \bar{\beta}\rangle\alpha = \beta p_2\langle \mathrm{id}, \bar{\beta}\rangle\alpha = \beta\bar{\beta}a = \beta$. But $\alpha \neq \beta$. Therefore, there do not exist $f, g : X^2 \to A$ with $(X, \Delta_x) = \mathrm{equ}(f, g)$, and so $\Delta_x \notin M$. □

Corollary 3.7. *Let $\mathscr{C}$ be a connected topological category, and $\mathscr{A}$ an extremal-epireflective subcategory with the property that the $\mathscr{A}$-regular morphisms are closed under composition. Then $X \in \mathrm{ob}\,\mathscr{A}$ iff Δ_x is an $\mathscr{A}$-regular morphism.*

Proof. This follows from Theorem 3.6 and the fact that $\Delta_x = \mathrm{Equ}(\pi_1, \pi_2)$. □

Corollary 3.8. *Under the assumptions of Corollary* 3.7, *there exists a (strong) factorization structure (E, M) on $\mathscr{C}$ with $\mathscr{A} = \mathscr{C}_M$.*

Proof. Let M denote the class of all $\mathscr{A}$-regular morphisms. Lemma 3.2 and Theorem 1.6 imply that M is the right factor of a strong factorization structure (E, M). From Corollary 3.7, we have that $X \in \mathrm{ob}\,\mathscr{A}$ iff $\Delta_X \in M$. □

Remark. The result in Theorem 3.6 holds in many categories that are not topological. For example, the existence of constant morphisms α and β with the desired property is guaranteed in many initially structured categories, such as **Haus**.

Moreover, there are categories where the existence of α and β both constants is impossible, yet the result follows if only β is a constant. An example of this is **Grp**, where if $G \notin \text{ob}\,\mathcal{A}$, $r_G : G \to G$, then $\alpha : \text{Ker}(r_G) \to G$ is the inclusion, and $\beta : \text{Ker}(r_G) \to G$ is defined by $\beta(x) = e$ for all x in $\text{Ker}(r_G)$.

4. Some examples

4.1. $\mathscr{C} = \mathbf{Top}$, $\mathcal{A}$ is the full subcategory whose objects are the singleton spaces.

(a) m is an $\mathcal{A}$-regular morphism iff m is an isomorphism;
(b) Every morphism is an $\mathcal{A}$-epi;
(c) The $\mathcal{A}$-regular morphisms are closed under composition;
(d) ($\mathcal{A}$-epi, $\mathcal{A}$-regular) is a factorization structure on **Top**. $X \in \text{ob}\,\mathcal{A}$ iff $\Delta_x \in$ $\mathcal{A}$-regular.

4.2 [9]. $\mathscr{C} = \mathbf{Top}$, $\mathcal{A} = \mathbf{FH}$, the full subcategory whose objects are functionally Hausdorff spaces. (Recall that X is functionally Hausdorff iff for all $x, y \in X$, $x \neq y$, there exists a continuous function $f: X \to [0, 1]$ such that $f(x) = 0$, $f(y) = 1$.

(a) $m : S \to X$ is a regular morphism in **FH** iff m is a closed embedding with the property that if $x \notin m(S)$, there exists $f: X \to [0, 1]$ such that $f(m(s)) = 0$ for all $s \in S$, $f(x) = 1$;
(b) $e : X \to Y \in \mathbf{FH}$-epi iff whenever $f: Y \to I$, fe constant$\Rightarrow f$constant;
(c) the **FH**-regular morphisms are closed under composition;
(d) (**FH**-epi, M') is a factorization structure on **Top**, where M' is the class of embeddings $m : X \to Y$ that satisfy the following condition: if $m(X) \subsetneq S \subseteq Y$, there exists $f: S \to [0, 1]$ such that $f(m(x)) = 0$ for all $x \in X$, $f(s) \neq 0$ for some $s \in S$.
(e) $\mathbf{FH} \neq \mathscr{C}_{M'}$.
(f) There is no factorization structure (E, M) on **Top** with $\mathbf{FH} = \mathscr{C}_M$.

Proof. The proofs of (a), (b) and (d) are straightforward. For (c) we will show that the regular morphisms in **FH** are not closed under composition.

Let R be the real line, and let T denote the Smirnov topology. T is defined as follows: T_1 is the Euclidean topology on R. $A = \{1/n: n = 1, 2, 3 \ldots\}$. $V \in T$ iff $V = U - B$, where $U \in T_1$, $B \subseteq A$. Thus every open set (in T) that contains A intersects every open set that contains 0. So there is no $f: R \to I$ with $f(a) = 0$ for all $a \in A$, $f(0) = 1$, and therefore the inclusion $A \to R$ is not regular in **FH**. However, the inclusions $A \to A \cup \{0\}$ and $A \cup \{0\} \to R$ are both regular in **FH**.

To see that $FH \neq \mathscr{C}_{M'}$, let X be a Hausdorff space that contains exactly two points that cannot be separated by a continuous function into $[0, 1]$. Then $\Delta_x \notin M'$.

To show that there is no factorization structure (E, M) on **Top** with $\mathbf{FH} = \mathscr{C}_M$, recall that $E \subseteq \mathscr{C}_M$-epi. Thus, if $\mathbf{FH} = \mathscr{C}_M$, $E \subseteq \mathbf{FH}$-epi, and therefore $M' \subseteq M$. The example mentioned in the preceding paragraph shows that there is a topological space $X \notin \text{ob } \mathbf{FH}$ with $\Delta_x \in M'$, and therefore $\Delta_x \in M$ whenever $E \subseteq \mathbf{FH}$-epi. $\square$

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